

Existence and exponential stability of a damped wave equation with dynamic boundary conditions and a delay term

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Abstract

In this paper we consider a multi-dimensional wave equation with dynamic boundary conditions related to the Kelvin-Voigt damping and a delay term acting on the boundary. If the weight of the delay term in the feedback is less than the weight of the term without delay or if it is greater under an assumption between the damping factor, and the difference of the two weights, we prove the global existence of the solutions. Under the same assumptions, the exponential stability of the system is proved using an appropriate Lyapunov functional. More precisely, we show that even when the weight of the delay is greater than the weight of the damping in the boundary conditions, the strong damping term still provides exponential stability for the system.

Keywords: Damped wave equations, boundary delay; global solutions; exponential stability; Kelvin-Voigt damping; dynamic boundary conditions.

1 Introduction

In this paper we consider the following linear damped wave equation with dynamic boundary conditions and a delay boundary term:

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u - \alpha \Delta u_t = 0, & x \in \Omega, \ t > 0, \\ u(x, t) = 0, & x \in \Gamma_0, \ t > 0, \\ u_{tt}(x, t) = - \left(\frac{\partial u}{\partial \nu}(x, t) + \frac{\alpha \partial u_t}{\partial \nu}(x, t) + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) \right) & x \in \Gamma_1, \ t > 0, \\ u(x, 0) = u_0(x) & x \in \Omega, \\ u_t(x, 0) = u_1(x) & x \in \Omega, \\ u_t(x, t - \tau) = f_0(x, t - \tau) & x \in \Gamma_1, \ t \in (0, \tau), \end{array} \right. \quad (1)$$

where $u = u(x, t)$, $t \geq 0$, $x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of \mathbb{R}^N , ($N \geq 1$), $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $mes(\Gamma_0) > 0$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative, α , μ_1 and μ_2 are positive constants. Moreover, $\tau > 0$ represents the time delay and u_0 , u_1 , f_0 are given functions belonging to suitable spaces that will be precised later.

This type of problems arise (for example) in modelling of longitudinal vibrations in a homogeneous bar in which there are viscous effects. The term Δu_t , indicates that the stress is proportional not only to the strain,

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but also to the strain rate. See [5]. From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such type of boundary conditions are usually called *dynamic boundary conditions*. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represents the Newton's law for the attached mass, (see [4, 1, 6] for more details). In the two dimension space, as showed in [24] and in the references therein, these boundary conditions arise when we consider the transverse motion of a flexible membrane Ω whose boundary may be affected by the vibrations only in a region. Also some dynamic boundary conditions as in problem (1) appear when we assume that Ω is an exterior domain of \mathbb{R}^3 in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [2] for more details). This type of dynamic boundary conditions are known as acoustic boundary conditions.

In the absence of the delay term (i.e. $\mu_2 = 0$) problem (1) has been investigated by many authors in recent years (see, e.g., [12], [13], [14], [15], [21], [22]).

Among the early results dealing with the *dynamic boundary conditions* are those of Grobbelaar-Van Dalsen [7, 8] in which the author has made contributions to this field.

In [7] the author introduced a model which describes the damped longitudinal vibrations of a homogeneous flexible horizontal rod of length L when the end $x = 0$ is rigidly fixed while the other end $x = L$ is free to move with an attached load. This yields to a system of two second order equations of the form

$$\begin{cases} u_{tt} - u_{xx} - u_{txx} = 0, & x \in (0, L), t > 0, \\ u(0, t) = u_t(0, t) = 0, & t > 0, \\ u_{tt}(L, t) = -[u_x + u_{tx}](L, t), & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = v_0(x), & x \in (0, L), \\ u(L, 0) = \eta, \quad u_t(L, 0) = \mu. \end{cases} \quad (2)$$

By rewriting problem (2) within the framework of the abstract theories of the so-called B -evolution theory, an existence of a unique solution in the strong sense has been shown. An exponential decay result was also proved in [8] for a problem related to (2), which describe the weakly damped vibrations of an extensible beam. See [8] for more details.

Subsequently, Zang and Hu [27], considered the problem

$$u_{tt} - p(u_x)_{xt} - q(u_x)_x = 0, \quad x \in (0, 1), t > 0$$

with

$$u(0, t) = 0, \quad p(u_x)_t + q(u_x)(1, t) + ku_{tt}(1, t) = 0, t \geq 0.$$

By using the Nakao inequality, and under appropriate conditions on p and q , they established both exponential and polynomial decay rates for the energy depending on the form of the terms p and q .

It is clear that in the absence of the delay term and for $\mu_1 = 0$, problem (2) is the one dimensional model of (1). Similarly, and always in the absence of the delay term, Pellicer and Solà-Morales [22] considered the one dimensional problem of (1) as an alternative model for the classical spring-mass damper system, and by using the dominant eigenvalues method, they proved that their system has the classical second order differential equation

$$m_1 u''(t) + d_1 u'(t) + k_1 u(t) = 0,$$

as a limit, where the parameter m_1 , d_1 and k_1 are determined from the values of the spring-mass damper system. Thus, the asymptotic stability of the model has been determined as a consequence of this limit. But they did not obtain any rate of convergence. See also [21, 23] for related results.

Recently, the present authors studied in [13] and [12] a more general situation of (1). They considered problem (1) with $\mu_2 = 0$, a nonlinear damping of the form $g(u_t) = |u_t|^{m-2} u_t$ instead of $\mu_1 u_t$ and a nonlinear source term $f(u) = |u|^{p-2} u$ in the right hand side of the first equation of problem (1). A local existence result was obtained by combining the Faedo-Galerkin method with the contraction mapping

theorem. Concerning the asymptotic behavior, the authors showed that the solution of such problem is unbounded and grows up exponentially when time goes to infinity if the initial data are large enough and the damping term is nonlinear. The blow up result was shown when the damping is linear. Also, we proved in [12] that under some restrictions on the exponents m and p , we can always find initial data for which the solution is global in time and decay exponentially to zero.

The main difficulty of the problem considered is related to the non ordinary boundary conditions defined on Γ_1 . Very little attention has been paid to this type of boundary conditions. We mention only a few particular results in the one dimensional space [16, 22, 11, 17].

The purpose of this paper is to study problem (1), in which a delay term acted in the dynamic boundary conditions. In recent years one very active area of mathematical control theory has been the investigation of the delay effect in the stabilization of hyperbolic systems and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [10] for more details).

In [19], Nicaise and Pignotti examined the wave equation with a linear boundary damping term with a delay. Namely, they looked to the following problem:

$$u_{tt} - \Delta u = 0, \quad x \in \Omega, t > 0, \quad (3)$$

where Ω is a bounded domain with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. On Γ_0 , they considered the Dirichlet boundary conditions. While on Γ_1 they assumed the following boundary conditions:

$$\frac{\partial u}{\partial \nu}(x, t) = \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau), \quad x \in \Gamma_1, t > 0. \quad (4)$$

They proved under the assumption

$$\mu_2 < \mu_1 \quad (5)$$

that the solution is exponentially stable. On the contrary, if (5) does not hold, they found a sequence of delays for which the corresponding solution of (3) will be unstable. The main approach used in [19], is an observability inequality obtained with a Carleman estimate. The same results were showed if both the damping and the delay are acting in the domain. We also recall the result by Xu, Yung and Li [26], where the authors proved the same result as in [19] for the one space dimension by adopting the spectral analysis approach. We point out that problem (1) has been already studied by Nicaise and Pignotti in [20] for $\mu_1 = 0$ and a time-varying delay. They find the same condition as the one used in this paper when $\mu_1 = 0$ by a different way. However our result and our Lyapunov functional are slightly different here. (See Remark 3.2 for more details), and we want to point out that this paper may be viewed as a continuation of the work of Nicaise and Pignotti [20] in which an additional damping term acts on the boundary and the study of the competition between these two damping terms is very interesting.

As it has been proved by Datko [9, Example 3.5], systems of the form

$$\begin{cases} w_{tt} - w_{xx} - aw_{xxt} = 0, & x \in (0, 1), t > 0, \\ w(0, t) = 0, \quad w_x(1, t) = -kw_t(1, t - \tau), & t > 0, \end{cases} \quad (6)$$

where a, k and τ are positive constants become unstable for an arbitrarily small values of τ and any values of a and k . In (6) and even in the presence of the strong damping $-aw_{xxt}$, without any other damping, the overall structure can be unstable. This was one of the main motivations for considering problem (1). (Of course the structure of problem (1) and (6) are different due to the nature of the boundary conditions in each problem).

Subsequently, Datko et al [10] treated the following one dimensional problem:

$$\begin{cases} u_{tt}(x, t) - u_{xx}(x, t) + 2au_t(x, t) + a^2u(x, t) = 0, & 0 < x < 1, t > 0, \\ u(0, t) = 0, & t > 0, \\ u_x(1, t) = -ku_t(1, t - \tau), & t > 0, \end{cases} \quad (7)$$

which models the vibrations of a string clamped at one end and free at the other end, where $u(x, t)$ is the displacement of the string. Also, the string is controlled by a boundary control force (with a delay) at the free end. They showed that, if the positive constants a and k satisfy

$$k \frac{e^{2a} + 1}{e^{2a} - 1} < 1,$$

then the delayed feedback system (7) is stable for all sufficiently small delays. On the other hand if

$$k \frac{e^{2a} + 1}{e^{2a} - 1} > 1,$$

then there exists a dense open set D in $(0, \infty)$ such that for each $\tau \in D$, system (7) admits exponentially unstable solutions.

As a consequence of what we have said before, two main questions naturally arise here:

- Is it possible for the damping term $-\Delta u_t$ to stabilize system (1) when the weight of the delay is greater than the weight of the boundary damping (i.e. when $\mu_2 \geq \mu_1$)?
- Does the particular structure of the problem prevents the instability result obtained in [9] for problem (6)?

One of the main purpose of this paper is to give positive answers to the above two questions. More precisely, we study the asymptotic behavior (as $t \rightarrow \infty$) and related decay rates for the corresponding solutions of system (1) where the question to be addressed here is whether the delay term $\mu_2 u_t(x, t - \tau)$ can destroy the stability of the system, which is exponentially stable in the absence of that delay [12]. As we shall see below, the presence of the strong damping term $\alpha \Delta u_t$ in (1) plays a decisive role in the stability of the whole system if (5) does not hold. Thanks to the energy method, we built appropriate Lyapunov functionals lead to stability results.

The paper is organized as follows: in the next section, we prove the global existence of the solutions by using the Lumer-Phillips' theorem in the same way as in [19]. In section 3, we show that if the weight of the delay is less than the weight of the damping, then the energy defined by (38) decays exponentially to zero. We also prove that even if the weight of the delay is greater than the weight of the damping, the solution still decays to zero exponentially provided that the damping parameter α satisfies an appropriate condition. Let us mention that without the damping factor α , Nicaise and Pignotti [19] proved the instability of the null stationary solution in the case $\mu_2 \geq \mu_1$, whereas we will show that if $\mu_2 \geq \mu_1$, by adding a condition of the form $\alpha > (\mu_2 - \mu_1)B^2$ (with B a constant defined later), we are able to prove the stability of the null stationary state thanks to a suitable choice of a Lyapunov function.

2 Well-posedness of Problem (1).

In this section we will first transform the delay boundary conditions by adding a new unknown. Then as in [19], we will use the Lumer-Phillips' theorem to prove the existence and uniqueness of the solution of problem (1).

2.1 Setup and notations

We denote

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) / u_{\Gamma_0} = 0\}.$$

We set γ_1 the trace operator from $H_{\Gamma_0}^1(\Omega)$ on $L^2(\Gamma_1)$ and $H^{1/2}(\Gamma_1) = \gamma_1(H_{\Gamma_0}^1(\Omega))$. We denote by B the norm of γ_1 namely:

$$\forall u \in H_{\Gamma_0}^1(\Omega), \|u\|_{2, \Gamma_1} \leq B \|\nabla u\|_2.$$

We recall that $H^{1/2}(\Gamma_1)$ is dense in $L^2(\Gamma_1)$ (see [18]).

We denote $E(\Delta, L^2(\Omega)) = \{u \in H^1(\Omega) \text{ such that } \Delta u \in L^2(\Omega)\}$ and recall that for a function $u \in E(\Delta, L^2(\Omega))$, $\frac{\partial u}{\partial \nu} \in H^{-1/2}(\Gamma_1)$ and the next Green's formula is valid (see [18]):

$$\int_{\Omega} \nabla u(x) \nabla v(x) dx = \int_{\Omega} -\Delta u(x) v(x) dx + \left\langle \frac{\partial u}{\partial \nu}; v \right\rangle_{\Gamma_1}, \quad \forall v \in H_{\Gamma_0}^1(\Omega), \quad (8)$$

where $\langle \cdot; \cdot \rangle_{\Gamma_1}$ means the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$.

By (\cdot, \cdot) we denote the scalar product in $L^2(\Omega)$ i.e. $(u, v) = \int_{\Omega} u(x) v(x) dx$. Also we mean by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$, and by $\|\cdot\|_{q, \Gamma_1}$ the $L^q(\Gamma_1)$ norm.

Throughout the paper, we use the standard notations as in the book [3] for example.

In order to prove the local existence of the solution of problem (1), we consider the following two cases :

case 1: $\mu_2 < \mu_1$. We may define a positive real number ξ such that:

$$\tau \mu_2 \leq \xi \leq \tau (2\mu_1 - \mu_2). \quad (9)$$

case 2: $\mu_2 \geq \mu_1$. We will suppose that the damping parameter α verifies:

$$\alpha > (\mu_2 - \mu_1) B^2. \quad (10)$$

In this case, we may define a positive real number ξ satisfying the two inequalities:

$$\xi \geq \tau \mu_2, \quad (11)$$

$$\alpha > \left(\frac{\mu_2}{2} + \frac{\xi}{2\tau} - \mu_1 \right) B^2 > 0. \quad (12)$$

2.2 Semigroup formulation of the problem

In this section, we prove the global existence and the uniqueness of the solution of problem (1). To overcome the problem of the boundary delay, we introduce, as in [19], the new variable:

$$z(x, \rho, t) = u_t(x, t - \tau \rho), \quad x \in \Gamma_1, \quad \rho \in (0, 1), \quad t > 0. \quad (13)$$

Then, we have

$$\tau z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, \quad \text{in } \Gamma_1 \times (0, 1) \times (0, +\infty). \quad (14)$$

Therefore, problem (1) is equivalent to:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = 0, & x \in \Omega, \quad t > 0, \\ \tau z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, & x \in \Gamma_1, \rho \in (0, 1), \quad t > 0, \\ u(x, t) = 0, & x \in \Gamma_0, \quad t > 0, \\ u_{tt}(x, t) = - \left(\frac{\partial u}{\partial \nu}(x, t) + \alpha \frac{\partial u_t}{\partial \nu}(x, t) + \mu_1 u_t(x, t) + \mu_2 z(x, 1, t) \right), & x \in \Gamma_1, \quad t > 0, \\ z(x, 0, t) = u_t(x, t), & x \in \Gamma_1, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u_t(x, 0) = u_1(x), & x \in \Omega, \\ z(x, \rho, 0) = f_0(x, -\tau \rho), & x \in \Gamma_1, \rho \in (0, 1). \end{cases} \quad (15)$$

The first natural question is the existence of solutions of the problem (15). In this section we will give a sufficient condition that guarantees the well-posedness of the problem.

For this purpose, as in [19], we will use a semigroup formulation of the initial-boundary value problem (15). If we denote $V := (u, u_t, \gamma_1(u_t), z)^T$, we define the energy space:

$$\mathcal{H} = H_{\Gamma_0}^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1 \times (0, 1)).$$

Clearly, \mathcal{H} is a Hilbert space with respect to the inner product

$$\langle V_1, V_2 \rangle_{\mathcal{H}} = \int_{\Omega} \nabla u_1 \cdot \nabla u_2 dx + \int_{\Omega} v_1 v_2 dx + \int_{\Gamma_1} w_1 w_2 d\sigma + \xi \int_{\Gamma_1} \int_0^1 z_1 z_2 d\rho d\sigma$$

for $V_1 = (u_1, v_1, w_1, z_1)^T$, $V_2 = (u_2, v_2, w_2, z_2)^T$ and ξ is defined by (9) or (11).

Therefore, if $V_0 \in \mathcal{H}$ and $V \in \mathcal{H}$, the problem (15) is formally equivalent to the following abstract evolution equation in the Hilbert space \mathcal{H} :

$$\begin{cases} V'(t) = \mathcal{A}V(t), & t > 0, \\ V(0) = V_0, \end{cases} \quad (16)$$

where $'$ denotes the derivative with respect to time t , $V_0 := (u_0, u_1, \gamma_1(u_1), f_0(., -\tau))^T$ and the operator \mathcal{A} is defined by:

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta u + \alpha \Delta v \\ -\frac{\partial u}{\partial \nu} - \alpha \frac{\partial v}{\partial \nu} - \mu_1 v - \mu_2 z(., 1) \\ -\frac{1}{\tau} z_{\rho} \end{pmatrix}.$$

The domain of \mathcal{A} is the set of $V = (u, v, w, z)^T$ such that:

$$(u, v, w, z)^T \in H_{\Gamma_0}^1(\Omega) \times H_{\Gamma_0}^1(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0, 1)), \quad (17)$$

$$u + \alpha v \in E(\Delta, L^2(\Omega)), \quad \frac{\partial(u + \alpha v)}{\partial \nu} \in L^2(\Gamma_1), \quad (18)$$

$$w = \gamma_1(v) = z(., 0) \text{ on } \Gamma_1. \quad (19)$$

The well-posedness of problem (15) is ensured by:

Theorem 2.1. *Suppose that $\mu_2 \geq \mu_1$ and $\alpha > (\mu_2 - \mu_1)B^2$ or $\mu_2 < \mu_1$. Let $V_0 \in \mathcal{H}$, then there exists a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ of problem (16). Moreover, if $V_0 \in \mathcal{D}(\mathcal{A})$, then*

$$V \in C(\mathbb{R}_+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H}).$$

Proof. To prove Theorem 2.1, we use the Lumer-Phillips' theorem. For this purpose, we show firstly that the operator \mathcal{A} is dissipative. Indeed, let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$. We have

$$\begin{aligned} \langle \mathcal{A}V, V \rangle_{\mathcal{H}} &= \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} v (\Delta u + \alpha \Delta v) dx \\ &\quad + \int_{\Gamma_1} w \left(-\frac{\partial u}{\partial \nu} - \alpha \frac{\partial v}{\partial \nu} - \mu_1 v - \mu_2 z(\sigma, 1) \right) d\sigma - \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z z_{\rho} d\rho d\sigma. \end{aligned}$$

But since $u + \alpha v \in E(\Delta, L^2(\Omega))$ and $\frac{\partial(u + \alpha v)}{\partial \nu} \in L^2(\Gamma_1)$, we may apply Green's formula (8) where the duality pairing $\langle \cdot, \cdot \rangle_{\Gamma_1}$ is simply the $L^2(\Gamma_1)$ inner product (because $w = \gamma_1(v) \in L^2(\Gamma_1)$) and obtain:

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = -\mu_1 \int_{\Gamma_1} w^2 d\sigma - \mu_2 \int_{\Gamma_1} z(\sigma, 1) w d\sigma - \alpha \int_{\Omega} |\nabla v|^2 dx - \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_{\rho} z d\rho dx. \quad (20)$$

At this point, we have to distinguish the following two cases:

Case 1: We suppose that $\mu_2 < \mu_1$. Let us choose then ξ that satisfies inequality (9). Using Young's inequality, (20) leads to

$$\begin{aligned} \langle \mathcal{A}V, V \rangle_{\mathcal{H}} + \alpha \int_{\Omega} |\nabla v|^2 dx + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_{\Gamma_1} w^2 d\sigma \\ + \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_{\Gamma_1} z^2(\sigma, 1) d\sigma \leq 0. \end{aligned}$$

Consequently, by using (9), we deduce that

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} \leq 0. \quad (21)$$

Case 2: We suppose that $\mu_2 \geq \mu_1$ and $\alpha > (\mu_2 - \mu_1)B^2$. Let us choose then ξ that satisfies the two inequalities (11) and (12). Using Young's inequality and the definition of the constant B , we can again prove that the inequality (21) holds. This means that in both cases \mathcal{A} is dissipative.

Now we show that $\lambda I - \mathcal{A}$ is surjective for all $\lambda > 0$.

For $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A})V = F,$$

which is:

$$\lambda u - v = f_1, \quad (22)$$

$$\lambda v - \Delta(u + \alpha v) = f_2, \quad (23)$$

$$\lambda w + \frac{\partial(u + \alpha v)}{\partial \nu} + \mu_1 v + \mu_2 z(\cdot, 1) = f_3, \quad (24)$$

$$\lambda z + \frac{1}{\tau} z_\rho = f_4. \quad (25)$$

To find $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of the system (22), (23), (24) and (25), we proceed as in [19]. Suppose u is determined with the appropriate regularity. Then from (22), we get:

$$v = \lambda u - f_1. \quad (26)$$

Therefore, from the compatibility condition on Γ_1 , (19), we determine $z(\cdot, 0)$ by:

$$z(x, 0) = v(x) = \lambda u(x) - f_1(x), \quad \text{for } x \in \Gamma_1. \quad (27)$$

Thus, from (25), z is the solution of the linear Cauchy problem:

$$\begin{cases} z_\rho = \tau(f_4(x) - \lambda z(x, \rho)), & \text{for } x \in \Gamma_1, \rho \in (0, 1), \\ z(x, 0) = \lambda u(x) - f_1(x). \end{cases} \quad (28)$$

The solution of the Cauchy problem (28) is given by:

$$z(x, \rho) = \lambda u(x)e^{-\lambda\rho\tau} - f_1e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau} \int_0^\rho f_4(x, \sigma)e^{\lambda\sigma\tau} d\sigma \quad \text{for } x \in \Gamma_1, \rho \in (0, 1). \quad (29)$$

So, we have at the point $\rho = 1$,

$$z(x, 1) = \lambda u(x)e^{-\lambda\tau} + z_1(x), \quad \text{for } x \in \Gamma_1 \quad (30)$$

with

$$z_1(x) = -f_1 e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^1 f_4(x, \sigma) e^{\lambda\sigma\tau} d\sigma, \quad \text{for } x \in \Gamma_1.$$

Since $f_1 \in H_{\Gamma_0}^1(\Omega)$ and $f_4 \in L^2(\Gamma_1) \times L^2(0, 1)$, then $z_1 \in L^2(\Gamma_1)$.

Consequently, knowing u , we may deduce v by (26), z by (29) and using (30), we deduce $w = \gamma_1(v)$ by (24).

In view of equations (23) and (24), we set, as in [20], $\bar{u} = u + \alpha v$. Then, from (26), we have

$$v = \lambda u - f_1 = \lambda(\bar{u} - \alpha v) - f_1.$$

Since $\lambda > 0$ and $\alpha > 0$, $1 + \lambda\alpha \neq 0$; thus we have:

$$v = \frac{\lambda}{1 + \lambda\alpha} \bar{u} - \frac{f_1}{1 + \lambda\alpha}. \quad (31)$$

But since $u = \bar{u} - \alpha v$, we have:

$$u = \frac{1}{1 + \lambda\alpha} \bar{u} + \frac{\alpha}{1 + \lambda\alpha} f_1. \quad (32)$$

From equations (23) and (24), \bar{u} must satisfy:

$$\frac{\lambda^2}{1 + \lambda\alpha} \bar{u} - \Delta \bar{u} = f_2 + \frac{\lambda}{1 + \lambda\alpha} f_1, \quad \text{in } \Omega \quad (33)$$

with the boundary conditions

$$\bar{u} = 0, \quad \text{on } \Gamma_0 \quad (34)$$

$$\frac{\partial \bar{u}}{\partial \nu} = f_3 - \lambda \gamma_1(v) - \mu_1 \gamma_1(v) - \mu_2 z(\cdot, 1), \quad \text{on } \Gamma_1 \quad (35)$$

the last equation at least formally since we don't have yet found the regularity of \bar{u} . Replacing u by its expression (32) and inserting it in equation (30), we get:

$$z(x, 1) = \frac{\lambda}{1 + \lambda\alpha} \bar{u}(x) e^{-\lambda\tau} + \frac{\lambda\alpha}{1 + \lambda\alpha} f_1(x) e^{-\lambda\tau} + z_1(x), \quad \text{for } x \in \Gamma_1.$$

Using the preceding expression of $z(\cdot, 1)$ and the expression of v given by (31), we have:

$$\frac{\partial \bar{u}}{\partial \nu} = - \frac{\lambda(\mu_2 e^{-\lambda\tau} + (\lambda + \mu_1))}{1 + \lambda\alpha} \bar{u} + f(x), \quad \text{for } x \in \Gamma_1 \quad (36)$$

with

$$f(x) = f_3(x) + \frac{(\lambda + \mu_1) - \mu_2 \lambda \alpha e^{-\lambda\tau}}{1 + \lambda\alpha} f_1(x) - \mu_2 z_1(x), \quad \text{for } x \in \Gamma_1.$$

From the regularity of f_1 , f_2 , z_1 , we get $f \in L^2(\Gamma_1)$.

The variational formulation of problem (33), (34), (36) is to find $\bar{u} \in H_{\Gamma_0}^1(\Omega)$ such that:

$$\begin{aligned} \int_{\Omega} \frac{\lambda^2}{1 + \lambda\alpha} \bar{u} \omega + \nabla \bar{u} \nabla \omega dx &+ \int_{\Gamma_1} \frac{\lambda(\mu_2 e^{-\lambda\tau} + (\lambda + \mu_1))}{1 + \lambda\alpha} \bar{u}(\sigma) \omega(\sigma) d\sigma, \\ &= \int_{\Omega} \left(f_2 + \frac{\lambda}{1 + \lambda\alpha} f_1 \right) \omega dx + \int_{\Gamma_1} f(\sigma) \omega(\sigma) d\sigma, \end{aligned} \quad (37)$$

for any $\omega \in H_{\Gamma_0}^1(\Omega)$. Since $\lambda > 0$, $\mu_1 > 0$, $\mu_2 > 0$, the left hand side of (37) defines a coercive bilinear form on $H_{\Gamma_0}^1(\Omega)$. Thus by applying the Lax-Milgram theorem, there exists a unique $\bar{u} \in H_{\Gamma_0}^1(\Omega)$ solution of (37).

Now, choosing $\omega \in \mathcal{C}_c^\infty$, \bar{u} is a solution of (33) in the sense of distribution and therefore $\bar{u} \in E(\Delta, L^2(\Omega))$. Thus using the Green's formula (8) in (37) and exploiting the equation (33) on Ω , we obtain finally:

$$\int_{\Gamma_1} \frac{\lambda(\mu_2 e^{-\lambda\tau} + (\lambda + \mu_1))}{1 + \lambda\alpha} \bar{u}(\sigma)\omega(\sigma) d\sigma + \left\langle \frac{\partial \bar{u}}{\partial \nu}; \omega \right\rangle_{\Gamma_1} = \int_{\Gamma_1} f(\sigma)\omega(\sigma) d\sigma \quad \forall \omega \in H_{\Gamma_0}^1(\Omega).$$

So $\bar{u} \in E(\Delta, L^2(\Omega))$ verifies (36) and by equation (32) and (31) we recover u and v and thus by (29), we obtain z and finally setting $w = \gamma_1(v)$, we have found $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of $(Id - \mathcal{A})V = F$.

Thus, the proof of Theorem 2.1, follows from the Lumer-Phillips' theorem. \square

3 Asymptotic behavior

3.1 Exponential stability for $\mu_2 < \mu_1$

In this subsection, we show that under the assumption $\mu_2 < \mu_1$, the solution of problem (15) decays to the null steady state with an exponential decay rate. For this goal, we use the energy method combined with the choice of a suitable Lyapunov functional.

For a positive constant ξ satisfying the strict inequality (9), (i.e. $<$ instead of \leq) we define the functional energy of the solution of problem (15) as

$$\begin{aligned} E(t) = E(t, z, u) &= \frac{1}{2} \left[\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 + \|u_t(t)\|_{2, \Gamma_1}^2 \right] \\ &+ \frac{\xi}{2} \int_{\Gamma_1} \int_0^1 z^2(\sigma, \rho, t) d\rho d\sigma \\ &= \frac{1}{2} E_1(t) + \frac{\xi}{2} \int_{\Gamma_1} \int_0^1 z^2(\sigma, \rho, t) d\rho d\sigma, \end{aligned} \quad (38)$$

where

$$E_1(t) = \|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 + \|u_t(t)\|_{2, \Gamma_1}^2.$$

Let us first remark that this energy is greater than the usual one of the solution of problem (1), namely $E_1(t)$.

Now, we prove that the above energy $E(t)$ is a decreasing function along the trajectories. More precisely, we have the following result:

Lemma 3.1. *Assume that $\mu_1 > \mu_2$, then the energy defined by (38) is a non-increasing positive function and there exists a positive constant C such that for (u, z) solution of (15), and for any $t \geq 0$, we have:*

$$\frac{dE(t)}{dt} \leq -C \left[\int_{\Gamma_1} u_t^2(\sigma, t) d\sigma + \int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma \right] - \alpha \int_{\Omega} |\nabla u_t(x, t)|^2 dx. \quad (39)$$

Proof. We multiply the first equation in (15) by u_t and perform integration by parts to get:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 + \|u_t(t)\|_{2, \Gamma_1}^2 \right] + \alpha \|\nabla u_t(t)\|_2^2 \\ &+ \mu_1 \|u_t(t)\|_{2, \Gamma_1}^2 + \mu_2 \int_{\Gamma_1} u_t(\sigma, t) u_t(\sigma, t - \tau) d\sigma = 0. \end{aligned} \quad (40)$$

We multiply the third equation in (15) by ξz , integrate the result over $\Gamma_1 \times (0, 1)$, we obtain:

$$\begin{aligned} \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_\rho z(\sigma, \rho, t) d\rho d\sigma &= \frac{\xi}{2\tau} \int_{\Gamma_1} \int_0^1 \frac{\partial}{\partial \rho} z^2(\sigma, \rho, t) d\rho d\sigma \\ &= \frac{\xi}{2\tau} \int_{\Gamma_1} (z^2(\sigma, 1, t) - z^2(\sigma, 0, t)) d\sigma. \end{aligned} \quad (41)$$

Using the definition (13) of z in the equality (40) and using the same technique as in the first step of the proof of Theorem 2.1, where we proved that \mathcal{A} is dissipative, inequality (39) holds. \square

The asymptotic stability result reads as follows:

Theorem 3.1. *Assume that $\mu_2 < \mu_1$. Then there exist two positive constants C and γ independent of t such that for (u, z) solution of problem (15), we have:*

$$E(t) \leq C e^{-\gamma t}, \quad \forall t \geq 0. \quad (42)$$

Proof. The proof of Theorem 3.1 relies on the construction of a Lyapunov functional.

For a small positive constant ε to be chosen later, we define:

$$\begin{aligned} L(t) = E(t) &+ \varepsilon \int_{\Omega} u(x, t) u_t(x, t) dx + \varepsilon \int_{\Gamma_1} u(\sigma, t) u_t(\sigma, t) d\sigma \\ &+ \frac{\varepsilon \alpha}{2} \int_{\Omega} |\nabla u(x, t)|^2 dx \\ &+ \varepsilon \xi \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) d\rho d\sigma. \end{aligned} \quad (43)$$

Let us say that the introduction of the last term in the Lyapunov functional L is inspired by the work of Nicaise and Pignotti [20].

It is straightforward to see that for $\varepsilon > 0$, $L(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and $\beta_2 > 0$ depending on ε such that for all $t \geq 0$

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t). \quad (44)$$

By taking the time derivative of the function L defined by (43), using the equations in problem (15), several integration by parts, and exploiting (39), we get:

$$\begin{aligned} \frac{dL(t)}{dt} \leq & - C \left[\int_{\Gamma_1} u_t^2(\sigma, t) d\sigma + \int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma \right] \\ & - \alpha \|\nabla u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 + \varepsilon \|u_t\|_2^2 + \varepsilon \|u_t\|_{2, \Gamma_1}^2 \\ & - \varepsilon \mu_1 \int_{\Gamma_1} u_t(\sigma, t) u(\sigma, t) d\sigma - \varepsilon \mu_2 \int_{\Gamma_1} z(\sigma, 1, t) u(\sigma, t) d\sigma \\ & + \varepsilon \xi \frac{d}{dt} \left(\int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) d\rho d\sigma \right). \end{aligned} \quad (45)$$

By using the second equation in (15), the last term in (45) can be treated as follows:

$$\begin{aligned} \varepsilon \xi \frac{d}{dt} \left(\int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) d\rho d\sigma \right) &= -\frac{2\varepsilon\xi}{\tau} \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z(\sigma, \rho, t) z_\rho(\sigma, \rho, t) d\rho d\sigma \\ &= -\frac{\varepsilon\xi}{\tau} \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} \frac{\partial}{\partial\rho} z^2(\sigma, \rho, t) d\rho d\sigma. \end{aligned}$$

Then, by using an integration by parts and the definition of z , the above formula can be rewritten as:

$$\begin{aligned} \varepsilon \xi \frac{d}{dt} \left(\int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) d\rho d\sigma \right) &= -\frac{\varepsilon\xi}{\tau} e^{-2\tau} \int_{\Gamma_1} z^2(1, \rho, t) d\rho d\sigma + \frac{\varepsilon\xi}{\tau} \int_{\Gamma_1} u_t^2(\sigma, t) d\sigma \\ &\quad - 2\varepsilon\xi \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) d\rho d\sigma. \end{aligned} \quad (46)$$

Applying Young's inequality, and the trace inequality, we obtain, for any $\delta > 0$:

$$\begin{aligned} \left| \int_{\Gamma_1} u_t u d\sigma \right| &\leq \delta \|u\|_{2,\Gamma_1}^2 + \frac{1}{4\delta} \|u_t\|_{2,\Gamma_1}^2 \\ &\leq \delta B^2 \|\nabla u\|_2^2 + \frac{1}{4\delta} \|u_t\|_{2,\Gamma_1}^2. \end{aligned} \quad (47)$$

Similarly, we have

$$\left| \int_{\Gamma_1} z(\sigma, 1, t) u(\sigma, t) d\sigma \right| \leq \frac{1}{4\delta} \int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma + \delta B^2 \|\nabla u\|_2^2. \quad (48)$$

Inserting (46), (47) and (48) into (45) and using Poincaré's inequality for u_t , in which we denote $C(\Omega)$ the Poincaré's constant, namely :

$$\forall w \in H_{\Gamma_0}^1(\Omega), \|w\|_2 \leq C(\Omega) \|\nabla w\|_2$$

we have:

$$\begin{aligned} \frac{dL(t)}{dt} &\leq - \left[C - \varepsilon \left(1 + \frac{\xi}{\tau} + \frac{\mu_1}{4\delta} \right) \right] \|u_t\|_{2,\Gamma_1}^2 \\ &\quad - \left[C - \varepsilon \left(\frac{\xi}{\tau} e^{-2\tau} + \frac{\mu_2}{4\delta} \right) \right] \int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma \\ &\quad - (\alpha - \varepsilon C(\Omega)^2) \|\nabla u_t\|_2^2 - \varepsilon (1 - B^2 \delta (\mu_1 + \mu_2)) \|\nabla u\|_2^2 \\ &\quad - 2\varepsilon \xi \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) d\rho d\sigma. \end{aligned} \quad (49)$$

We choose now δ small enough in (49) such that

$$\delta < \frac{1}{B^2 (\mu_1 + \mu_2)}.$$

Once δ is fixed, using once again Poincaré's inequality in (49), we may pick ε small enough to obtain the existence of $\eta > 0$, such that:

$$\frac{dL(t)}{dt} \leq -\eta \varepsilon E(t), \quad \forall t \geq 0. \quad (50)$$

On the other hand, by virtue of (44), setting $\gamma = -\eta \varepsilon / \beta_2$, the last inequality becomes:

$$\frac{dL(t)}{dt} \leq -\gamma L(t), \quad \forall t \geq 0. \quad (51)$$

Hence, integrating the previous differential inequality (51) between 0 and t , we get

$$L(t) \leq C_* e^{-\gamma t}, \quad \forall t \geq 0,$$

for some positive constant C_* .

Consequently, by using (44) once again, we conclude that it exists $C > 0$ such that:

$$E(t) \leq C e^{-\gamma t}, \quad \forall t \geq 0.$$

This completes the proof of Theorem 3.1 . □

3.2 Exponential stability for $\mu_2 > \mu_1$ and $\alpha > (\mu_2 - \mu_1)B^2$

As, we have said in the Introduction, and it is clearly observed in Theorem 3.1, that the strong internal damping compensates the destabilizing effect of the delay in the boundary condition.

In this section, we assume that $\mu_2 > \mu_1$ and $\alpha > (\mu_2 - \mu_1)B^2$. As we will see, we cannot directly perform the same proof as for the case where $\mu_2 \leq \mu_1$, since the boundary delay term $\mu_2 u_t(x, t - \tau)$ is greater than the normal one $\mu_1 u_t(x, t)$, i.e. ($\mu_2 \geq \mu_1$). So we have to control this term by the damping term $\alpha \Delta u_t$ in the equation.

Remark 3.1. *In the case 2, namely $\mu_2 > \mu_1$ the condition $\alpha > (\mu_2 - \mu_1)B^2$ permits us to find ξ satisfying (11)-(12). This choice of ξ is essential in the proofs of Lemma 3.2 and Theorem 3.2 below.*

Lemma 3.2. *Assume that $\mu_2 > \mu_1$ and $\alpha > (\mu_2 - \mu_1)B^2$. For any ξ satisfying (11)-(12), the energy defined by (38) is a non-increasing positive function and there exists a positive constant κ such that for (u, z) solution of (15), and for any $t \geq 0$, we have:*

$$\frac{dE(t)}{dt} \leq -\kappa \left[\int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma + \int_{\Omega} |\nabla u_t(x, t)|^2 dx \right]. \quad (52)$$

Proof. Let us first recall from (40) and (41) the following identity

$$\begin{aligned} \frac{dE(t)}{dt} &= -\alpha \int_{\Omega} |\nabla u_t|^2 dx - \left(\mu_1 - \frac{\xi}{2\tau} \right) \int_{\Gamma_1} u_t^2(\sigma, t) d\sigma \\ &\quad - \frac{\xi}{2\tau} \int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma - \mu_2 \int_{\Gamma_1} u_t(\sigma, t) z(\sigma, 1, t) d\sigma. \end{aligned} \quad (53)$$

Now, using Young's inequality, then (53) takes the form:

$$\begin{aligned} \frac{dE(t)}{dt} &\leq -\alpha \int_{\Omega} |\nabla u_t|^2 dx - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_{\Gamma_1} u_t^2(\sigma, t) d\sigma \\ &\quad - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma. \end{aligned} \quad (54)$$

Since,

$$\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} < 0,$$

then, using the trace inequality, we obtain:

$$\begin{aligned} \frac{dE(t)}{dt} &\leq \left(-\alpha - B^2 \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \right) \int_{\Omega} |\nabla u_t|^2 dx \\ &\quad - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_{\Gamma_1} z^2(\sigma, 1, t) d\sigma. \end{aligned}$$

Using the two inequalities (11)-(12) that ξ satisfy, we may find $\kappa > 0$ such that the inequality (52) holds. \square

We can now state that under the same assumption as in Lemma 3.2, the system (15) is also exponentially stable. The second stability result reads as follows:

Theorem 3.2. *Assume that $\mu_2 > \mu_1$ and $\alpha > (\mu_2 - \mu_1)B^2$. For any ξ satisfying (11)-(12), there exist two positive constants \overline{C} and $\overline{\gamma}$ independent of t such that for (u, z) solution of problem (15), we have:*

$$E(t) \leq \overline{C} e^{-\overline{\gamma}t}, \quad \forall t \geq 0. \quad (55)$$

Proof of Theorem 3.2. We use the same Lyapunov function as in the previous section, namely, for a small positive constant ε to be chosen later, we define:

$$\begin{aligned} L(t) = E(t) &+ \varepsilon \int_{\Omega} u(x, t) u_t(x, t) \, dx + \varepsilon \int_{\Gamma_1} u(\sigma, t) u_t(\sigma, t) \, d\sigma \\ &+ \frac{\varepsilon \alpha}{2} \int_{\Omega} |\nabla u(x, t)|^2 \, dx \\ &+ \varepsilon \xi \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) \, d\rho \, d\sigma. \end{aligned}$$

By taking the time derivative of the function L , using the equations in problem (15), several integration by parts, and exploiting (52), we get:

$$\begin{aligned} \frac{dL(t)}{dt} \leq & - \kappa \left[\int_{\Gamma_1} z^2(\sigma, 1, t) \, d\sigma + \int_{\Omega} |\nabla u_t(x, t)|^2 \, dx \right] \\ & - \alpha \|\nabla u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 + \varepsilon \|u_t\|_2^2 + \varepsilon \|u_t\|_{2, \Gamma_1}^2 \\ & - \varepsilon \mu_1 \int_{\Gamma_1} u_t(\sigma, t) u(\sigma, t) \, d\sigma - \varepsilon \mu_2 \int_{\Gamma_1} z(\sigma, 1, t) u(\sigma, t) \, d\sigma \\ & + \varepsilon \xi \frac{d}{dt} \left(\int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) \, d\rho \, d\sigma \right). \end{aligned} \quad (56)$$

Let us remark that equality (46), and inequalities (47) and (48) used in the the proof of Theorem 3.1 are still valid. Inserting (46), (47) and (48) into (56) and using Poincaré's inequality, we have:

$$\begin{aligned} \frac{dL(t)}{dt} \leq & - \left[\kappa - \varepsilon \left(\frac{\xi}{\tau} e^{-2\tau} + \frac{\mu_2}{4\delta} \right) \right] \int_{\Gamma_1} z^2(\sigma, 1, t) \, d\sigma \\ & - \left(\kappa - \varepsilon \left(B^2 + 1 + \frac{B^2 \xi}{\tau} + \frac{\mu_1}{4\delta} \right) \right) \|\nabla u_t\|_2^2 \\ & - \varepsilon (1 - C(\Omega)^2 \delta (\mu_1 + \mu_2)) \|\nabla u\|_2^2 \\ & - 2\varepsilon \xi \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} z^2(\sigma, \rho, t) \, d\rho \, d\sigma. \end{aligned} \quad (57)$$

The remaining part of the proof is similar to the one of the proof of Theorem 3.1: by choosing firstly δ and then ε , we may find $\overline{\gamma} > 0$ independent of t such that:

$$\frac{dL(t)}{dt} \leq -\overline{\gamma} L(t), \quad \forall t \geq 0.$$

This inequality permits us to conclude the proof of Theorem 3.2. □

Remark 3.2. After our work had been submitted, we noticed that a similar problem has been already studied by Nicaise and Pignotti [20]. However our problem is slightly different from the one considered in [20]:

- First, we have the extra term $\mu_1 u_t$ on the boundary conditions, which makes the stability analysis independent of the strong damping $\alpha \Delta u_t$, for $\mu_1 > \mu_2$. This is not the case in [20]. See the assumption (2.56) in [20]. Moreover, the paper [20] does not cover the case $\alpha = 0$ at all.
- Secondly, our Lyapunov functional is different from the one used in [20]. We choose to use this one to have a strong control of the boundary delay term.

Finally, let us remark that our assumption $(\mu_2 - \mu_1)B^2 < \alpha$ is exactly the same than the one obtained in [20, Condition (2.56)], when $\mu_1 = 0$. So this work can be viewed as a continuation of the works of Nicaise and Pignotti [20]

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